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SOME CONSIDERATIONS ON THE PROBLEM OF TORSION AND FLEXURE OF PRISMATICAL BEAMS†

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Abstract—Based on previous work for the problem of end-loaded cantilever beams with loading conditions prescribed in terms of displacements rather than stresses, for the purpose of defining shear center location in terms of influence coefficients, the present report uses the principles of minimum potential and complementary energy for the establishment of upper and lower bounds for these influence coefficients. In applying the complementary energy principle we modify an earlier procedure by not departing from a St. Venant stress distribution and by using instead a stress approximation in which both shear and normal stress distributions are determined through use of the variational equation. In doing this the problem is solved more simply than before, for a more general class of cases.

INTRODUCTION

Recent considerations of the problem of the end-loaded cantilever beam, with the conditions of loading prescribed in terms of displacements rather than in terms of stresses, have lead to defining relations for shear center and twist center location in terms of influence coefficients, of a particularly simple nature[1, 2]. It was furthermore shown that approximate values of these influence coefficients, leading to approximate expressions for shear and twist center coordinates, could be obtained by using St. Venant-type torsional and flexural stress distributions in a Rayleigh–Ritz sense in conjunction with the principle of minimum complementary energy[1]. In what follows we extend these results in several directions.

We begin by making explicit the distinction between expressions for shear and twist center coordinates in terms of flexibility coefficients (which were previously considered) and in terms of stiffness coefficients (where it is shown that for the case of principal centroidal coordinate axes the final formulas are as simple as the formulas in terms of flexibility coefficients).

We supplement our earlier statement of a minimum complementary energy equation for the case of prescribed rigid-body type in-plane end section displacements[1] by a statement of the associated minimum potential energy equation. Furthermore, we consider, in addition to the case of prescribed in-plane end displacements, conditions of loading in a somewhat unconventional fashion, specifiying the form but not the magnitude of these displacements at the loaded end of the beam, and at the same time specifying the resultants but not the distribution of in-plane stresses at this end. We find that it is a simple matter to state a minimum potential energy equation in such a way as to apply to the latter case, but we leave open the question of an appropriate miminum complementary energy equation.

We use minimum potential and complementary energy equation statements for the establishment of upper and lower bound relations for quadratic forms involving flexibility and stiffness coefficients, respectively. We give both types of bounds for stiffness coefficients, but a lower bound only for flexibility coefficients, pending formulation of a minimum complementary energy equation for the mixed displacement-stress boundary condition case described above.

In applying the principle of minimum potential energy for the approximate determination of flexibility and stiffness coefficients we utilize displacement approximations which have previously been used for the analysis of the problems of torsion and flexure with end-section restraint against warping[3–5]. We think that it has not previously been recognized that these approximations may be utilized, beyond allowing assessments of the effect of end section restraint, for the purpose of deducing bound relations for coefficients entering into the formulas for shear and twist center coordinates.

In applying the principle of minimum complementary energy we significantly modify our earlier procedure[1] by *not* departing from a St. Venant-type shear distribution. Instead we

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utilize an approximation in which the distributions of both shear *and* normal stress over the cross section are determined through use of the variational equation. In so doing the problem is, without special effort, solved for a more general class of cases than heretofore, in a manner which is thought to represent a significant simplification of the earlier work.

Remarkably, the approximate results for flexibility and stiffness coefficients which are obtained on the basis of making quite disimilar approximative assumptions in connection with the use of the minimum potential energy equation and of the minimum complementary energy equation lead to identical approximate expressions for the coordinates of the center of shear and of twist. This, in conjunction with our upper and lower bound relations, leads to the conclusion that our approximate results are in fact exact, in the limit of vanishing a/L, where a is a representative cross sectional width dimension and L is the axial length of the beam, with the possibility left open to refine the analysis so as to account for the (generally small) effect of finite values of a/L.

A FORMULATION OF THE PROBLEMS OF TORSION AND FLEXURE

We consider a body with boundaries defined by a cylindrical surface $f(x, y) = 0$ and two planes $z = 0$ and $z = L$. We designate displacements by u, v, w and stresses by σ_x, τ_{xy}, etc. and we assume that the normal three-dimensional *homogeneous* equations of *linear* elasticity hold. We further assume that the boundary portion $f = 0$ is traction free and that the boundary portion $z = 0$ is fixed.

In regard to the boundary portion $z = L$ we assume the absence of normal tractions, in conjunction with a rigid body translation and rotation distribution of tangential displacements, i.e. we stipulate the conditions

$$z = L; \quad \sigma_z = 0, \quad u = U - y\Theta, \quad v = V + x\Theta. \tag{1}$$

We now observe that in writing eqn (1) we may, or we may not, stipulate additionally the magnitudes of U, V, Θ. If we do, as we have done earlier in conjunction with applications of the principle of minimum complementary energy[1, 2], then eqn (1) is a complete statement of loading conditions. If we do not and leave the magnitudes of U, V, Θ unspecified then we must, in order to complete the statement of loading conditions, prescribe additionally the magnitude of two transverse force components P, Q, and of an axial torque T, as follows

$$z = L: \qquad \int(\tau_{xz}, \tau_{yz})\,\mathrm{d}S = (P, Q), \qquad \int(\tau_{yz}x - \tau_{xz}y)\,\mathrm{d}S = T. \tag{2}$$

In view of the linearity and homogeneity of the problem we have that P, Q and T will be combinations of U, V and Θ, and vice versa, in the form

$$\begin{aligned} &P = K_{PU}U + K_{PV} + K_{P\Theta}\Theta, \\ &Q = K_{QU}U + \ldots, \quad T = K_{TU}U + \ldots + K_{T\Theta}\Theta, \end{aligned} \tag{3}$$

and

$$\begin{aligned} &U = C_{UP}P + C_{UQ}Q + C_{UT}T, \\ &V = C_{VP}P + \ldots, \quad \Theta = C_{\Theta P}P + \ldots + C_{\Theta T}T, \end{aligned} \tag{4}$$

with the K_{PU}, etc. being *stiffness* coefficients, the C_{UP}, etc. being *flexibility* coefficients, and with the expectation of symmetry for the matrix of the K's as well as for the matrix of the C's.

Having eqns (3) and (4) we obtain the coordinates of the *center of twist*, x_T, y_T, as the coordinates x, y of that point in the end cross section for which $u = v = 0$ in eqn (1) while at the same time $P = Q = 0$, i.e. in the form

$$y_T = (U/\Theta)_{P=Q=0}, \quad x_T = -(V/\Theta)_{P=Q=0}, \tag{5}$$

and we obtain the coordinates of the *center of shear*, x_S, y_S, as the coordinates of the point of intersection of the lines of action of the forces P, Q for the case $\Theta = 0$, in conjunction with the torque T being solely due to the forces P, Q, that is, upon setting in eqns (3) or (4)

$$\Theta = 0,\; T = Qx_S - Py_S. \tag{6}$$

It turns out that with these defining relations the simpler form of the results appears through use of the flexibility coefficients, namely

$$y_T = \frac{C_{UT}}{C_{\Theta T}},\quad y_S = \frac{C_{\Theta P}}{C_{\Theta T}},\quad x_T = -\frac{C_{VT}}{C_{\Theta T}},\quad x_S = -\frac{C_{\Theta Q}}{C_{\Theta T}} \tag{7}$$

with $y_T = y_S$ and $x_T = x_S$ for the normal case of a symmetric C-matrix. The corresponding relations in terms of the coefficients K in eqn (3) come out to be ratios of certain second order minors of the third order determinant of the coefficient matrix in (3). It will be useful to note for what follows that we have, on the basis of eqns (5) and (3),

$$y_T = \frac{K_{PV}K_{Q\Theta} - K_{QV}K_{P\Theta}}{K_{PU}K_{QV} - K_{QU}K_{PV}},\quad x_T = \frac{K_{PU}K_{Q\Theta} - K_{QU}K_{P\Theta}}{K_{PU}K_{QV} - K_{QU}K_{PV}} \tag{8}$$

again with $y_S = y_T$ and $x_S = x_T$ in normal circumstances, and with the important special-case formulas

$$y_T = -\frac{K_{P\Theta}}{K_{PU}},\quad x_T = \frac{K_{Q\Theta}}{K_{QV}}, \tag{9}$$

which result upon assuming that $K_{PV} = K_{QU} = 0$.

MINIMUM COMPLEMENTARY AND POTENTIAL ENERGY EQUATIONS FOR THE PROBLEMS OF TORSION AND FLEXURE

We now assume that the material of the beam is such that its stress–strain relations may be written in the alternative forms

$$\sigma_x = \partial A/\partial \epsilon_x,\quad \tau_{xy} = \partial A/\partial \gamma_{xy}, \ldots, \tag{10a}$$

and

$$\epsilon_x = \partial B/\partial \sigma_x,\quad \gamma_{xy} = \partial B/\partial \tau_{xy}, \ldots. \tag{10b}$$

We then have for the case of *prescribed* $U = \bar{U}$, $V = \bar{V}$, $\Theta = \bar{\Theta}$ as minimum *complementary* energy condition the variational equation $\delta I_s = 0$, where

$$I_s = -\iint B \,\mathrm{d}S\,\mathrm{d}z + \bar{U}P + \bar{V}Q + \bar{\Theta}T. \tag{11a}$$

In (11a) the stresses σ_x, τ_{xy}, etc. must satisfy the differential equations of equilibrium and all stress boundary conditions, and the variational equation is equivalent to all strain displacement relations and displacement boundary conditions. We have earlier considered the application of this variational problem for the approximate determination of flexibility coefficients in conjunction with stress distributions corresponding to the solutions of the St. Venant torsion and flexure problem [1, 2].

For an alternate formulation of the problem, within the context of the principle of minimum *potential* energy, which we have not considered previously, we now prescribe $P = \bar{P}$, $Q = \bar{Q}$, $T = \bar{T}$, in association with the end displacement distribution (1). We then have that the appropriate form of the principle of minimum potential energy is the variational equation $\delta I_d = 0$ where

$$I_d = \iint A \,\mathrm{d}S\,\mathrm{d}z - \bar{P}U - \bar{Q}V - \bar{T}\Theta. \tag{11b}$$

In (11b) the strains ϵ_x, γ_{xy}, etc. are given in terms of displacement derivatives, the displacement components u, v, w must vanish for $z = 0$, and u, v must be as in eqn (1) for $z = L$, *with no restrictions imposed on* U, V, Θ. The variational problem as stated has as Euler equations the differential equations of equilibrium in the interior and all conditions of prescribed stress on the surface.

Having previously used the variational equation $\delta I_s = 0$ for the determination of *flexibility* coefficient approximations—without regard to the fact that the form of eqns (11a) and (3) indicates that it would be more natural to use this relation for the determination of *stiffness* coefficients—we have not previously used the relation $\delta I_d = 0$ which, it is apparent from (11b) and (4), *is* a natural starting point for the approximate determination of flexibility coefficients.

Previous general considerations on upper and lower bound-determinations for influence coefficients[6] indicate that the use of I_s is associated with the possibility of determining *lower bounds for stiffness coefficients* K and that the use of I_d gives the possibility of determining *lower bounds for flexibility coefficients* C.

Furthermore, we know that the use of a potential energy function I_d^* defined by

$$I_d^* = \iint A \, \mathrm{d}S \, \mathrm{d}z, \tag{12}$$

with I_d^* differing from I_d by the stipulation that in it $U = \bar{U}$, $V = \bar{V}$, $\Theta = \bar{\Theta}$, will be involved in the determination of *upper bounds for stiffness coefficients*.

In order to obtain upper bounds for flexibility coefficients we should have a counterpart I_s^* to I_s as defined in (11a), with $\bar{U}$, $\bar{V}$, $\bar{\Theta}$ replaced by $\bar{P}$, $\bar{Q}$, $\bar{T}$, in such a way that the *form* of the tangential end displacement distribution remains prescribed in accordance with eqn (1). We do not, at this time, know the way in which to introduce these "partial" displacement boundary conditions into the principle of minimum complementary energy. Because of this we do not here establish the appropriate form of I_s^* (which, if symmetry considerations were the principal guides, ought to be given by $-\iint B \, \mathrm{d}S \, dz$).

UPPER AND LOWER BOUND RELATIONS FOR INFLUENCE COEFFICIENTS

Appropriate transformations of the functionals I_s, I_d and I_d^*, as defined in eqns (11) and (12) lead to the upper and lower bound relations

$$\tilde{I}_s \leq \frac{1}{2}(\bar{U}P + \bar{V}Q + \bar{\Theta}T) \leq \tilde{I}_d^*, \tag{13}$$

$$-\frac{1}{2}(U\bar{P} + V\bar{Q} + \Theta\bar{T}) \leq \tilde{I}_d, \tag{14}$$

with the missing l.h.s. of eqn (14) making it evident that it would be useful to have a functional I_s^* as discussed at the end of the preceding section. In eqn (13) $\tilde{I}_s$ corresponds to the functional I_s in (11a), with any stresses $\tilde{\sigma}_z$, $\tilde{\tau}_{xy}$, etc. which satisfy equilibrium differential equations and stress boundary conditions, and $\tilde{I}_d^*$ corresponds to I_d^* in (12), with any differentiable displacement state $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ which satisfies the stipulated displacement boundary conditions, with the same rules connecting $\tilde{I}_d$ in (14) and I_d in (11b).

In order to see that (13) and (14) represent bound relations for stiffness and flexibility coefficients respectively, we observe that $\tilde{I}_d^*$ as well as $\tilde{I}_s$ will be quadratic forms in $\bar{U}$, $\bar{V}$ and $\bar{\Theta}$, which may be written as

$$\tilde{I}_d^* = \frac{1}{2} K_{UU}^U \bar{U}^2 + K_{UV}^U \bar{U}\bar{V} + \ldots + \frac{1}{2} K_{\Theta\Theta}^U \bar{\Theta}^2 \tag{15a}$$

$$\tilde{I}_s = \frac{1}{2} K_{UU}^L \bar{U}^2 + \ldots + \frac{1}{2} K_{\Theta\Theta}^L \bar{\Theta}^2. \tag{15b}$$

At the same time the quantity which is bounded from above and below in eqn (13) may be written, with the help of eqns (3), as

$$\bar{U}P + \bar{V}Q + \bar{\Theta}T = K_{PU}\bar{U}^2 + (K_{PV} + K_{QU})\,\bar{U}\bar{V} + \ldots + K_{T\Theta}\bar{\Theta}^2 \tag{16}$$†

where $K_{PU} = K_{UU}$, $1/2(K_{PV} + K_{QU}) = K_{PV} = K_{QU} \equiv K_{UV}$, etc. We note that when $\bar{V} = \bar{\Theta} = 0$ eqns (13), (15) and (16) specialize to a previously stated result for the problem of flexure, within the framework of the theory of plane stress[7]. Analogously, setting $\bar{U} = 0$ and $\bar{V} = 0$ gives a simple explicit result for the problem of twisting *with end restraint against warping*, which may not have been stated previously.‡ We further note the evident simplifications which occur in the above upon stipulating symmetry conditions which result in bound relations for quadratic forms in *two* variables in place of the relations for the general three-variable case.

We next consider eqn (14), written in the form $-\bar{I}_d \leq 1/2(U\bar{P} + V\bar{Q} + \Theta\bar{T})$. In this, eqn (11b) enables us to write with suitable coefficients C^L,

$$-\bar{I}_d = \frac{1}{2}C^L_{PP}\bar{P}^2 + C^L_{PQ}\bar{P}\bar{Q} + \ldots + \frac{1}{2}C^L_{TT}\bar{T}^2. \tag{17}$$

At the same time, we have, through the use of (4), that

$$\bar{P}U + \bar{Q}V + \bar{T}\Theta = C_{UP}\bar{P}^2 + (C_{UQ} + C_{VP})\bar{P}\bar{Q} + \ldots + C_{\Theta T}\bar{T}^2, \tag{18}$$

with the obvious consequences that

$$C^L_{PP} \leq C_{UP}, \quad C^L_{QQ} \leq C_{VQ}, \quad C^L_{TT} \leq C_{\Theta T}, \tag{19}$$

but with the determination of bounds for the coefficients $C_{UQ} + C_{VP}$, etc. of the mixed terms, and therewith of bounds for the coordinates of the centers of shear and of twist, depending upon the establishment of a bound functional I^*_s for the l.h.s. of eqn (14).

APPROXIMATE DETERMINATION OF SHEAR AND TWIST CENTER LOCATION THROUGH USE OF MINIMUM POTENTIAL ENERGY PRINCIPLE

Similar to what has been done in earlier work on problems of combined twisting and bending of beams[3–5] we begin by stipulating as approximations for components of displacement

$$\bar{u} = u(z) - y\theta(z), \quad \bar{v} = v(z) + x\theta(z), \tag{20}$$

$$\bar{w} = w_0(z) + x\alpha(z) + y\beta(z) + \phi(x, y)\lambda(z). \tag{21}$$

In this $\phi(x, y)$ is a function which is to be *assumed* suitably, with the various functions of z in (20) and (21) to be determined by the variational procedure, with or without imposition of additional constraint relations.

As regards stress–strain relations we here consider a material possessing a limiting type orthotropy, in such a way that there is partial rigidity, with each cross section translating and rotating as an entity.§ Our limiting-type stress–strain relations are three relations $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$, in conjunction with three relations of the form

$$\sigma_z = E\epsilon_z, \quad \tau_{xz} = G\gamma_{xz}, \quad \tau_{yz} = G\gamma_{yz}. \tag{22}$$

With (22) we have as expression for the strain energy function A in eqn (11)

$$2A = E\epsilon_z^2 + G\gamma_{xz}^2 + G\gamma_{yz}^2, \tag{23}$$

†After the manuscript of this paper had been completed, a letter by S. Nair informed the author of an independent and nearly simultaneous derivation of the result expressed by eqns (13), (15) and (16).

‡The result for twisting without restraint against warping is contained in an early fundamental paper by Trefftz[8].

§This assumption is meaningful for sufficiently slender beams only, where its approximate validity depends on the relative insignificance of the components of stress σ_x, σ_y, τ_{xy} in comparison with the components σ_z, τ_{xz}, τ_{yz}.

where, on the basis of (20) and (21),

$$\epsilon_z = w_0' + x\alpha' + y\beta' + \phi\lambda', \tag{24}$$

$$\gamma_{xz} = u' + \alpha - y\theta' + \lambda\phi_{,x}, \quad \gamma_{yz} = v' + \beta + x\theta' + \lambda\phi_{,y}. \tag{25}$$

In introducing (23) to (25) into the strain energy integral we shall assume that *the origin of the x, y system of axes is at the centroid of the cross section*, that is we stipulate the relations $\int(x, y)E\,\mathrm{d}S = 0$. Furthermore, we shall assume that *ϕ is the warping function for St. Venant torsion* of a homogeneous beam with the same cross section as the given beam, i.e. we shall assume that ϕ is determined through the relations $\nabla^2\phi = 0$, $\int(\phi_x - y,\ \phi_{,y} + x)\,\mathrm{d}S = (0, 0)$, $(-y + \phi_{,x})\,\mathrm{d}y - (x + \phi_{,y})\,\mathrm{d}x = 0$ along the boundary $f = 0$ of the cross section, we shall set as an abbreviation

$$D = \int (\phi_{,x}^2 + \phi_{,y}^2)\,\mathrm{d}S = \int (y\phi_{,x} - x\phi_{,y})\,\mathrm{d}S, \tag{26a}$$

and we shall stipulate, as we may, that $\int E\phi\,\mathrm{d}S = 0$. *To be consistent with our choice of ϕ we furthermore assume in what follows that $G = const.$* With this we now obtain as expression for the approximation $\tilde{I}_d$ to the functional I_d in eqn (11)

$$\tilde{I}_d = \frac{1}{2}\int\{S_E(w_0')^2 + I_{xx}(\alpha')^2 + I_{yy}(\beta')^2 + 2I_{xy}\alpha'\beta' + \Gamma(\lambda')^2 + 2\Gamma_x\alpha'\lambda' + 2\Gamma_y\beta'\lambda'$$
$$+ G[(u' + \alpha)^2 + (v' + \beta)^2 - 2(u' + \alpha)\theta'\int y\,\mathrm{d}S + 2(v' + \beta)\theta'\int x\,\mathrm{d}S$$
$$+ I_p(\theta')^2 + D\lambda^2 - 2D\lambda\theta']\}\,\mathrm{d}z - \bar{P}U - \bar{Q}V - \bar{T}\Theta, \tag{27}$$

where $U = u(L)$, $V = v(L)$, $\Theta = \theta(L)$, and

$$I_p = \int(x^2 + y^2)\,\mathrm{d}S, \quad (\Gamma, \Gamma_x, \Gamma_y) = \int(1, x, y)\phi E\,\mathrm{d}S,$$
$$(S_E, I_{xx}, I_{xy}, I_{yy}) = \int(1, x^2, xy, y^2)E\,\mathrm{d}S. \tag{26b}$$

Inasmuch as we are concerned with approximate rather than exact results, we shall now further assume that translational deflections due to transverse shear may be neglected and that the entire transverse shear strain energy is that due to twisting. Considering the form of (27) the desired reduction will be accomplished upon introducing the additional constraint relations

$$\alpha = -u', \quad \beta = -v', \quad \lambda = \theta'. \tag{28}$$

Anticipating furthermore the result $w_0 = 0$, we will then have in place of eqn (27)

$$\tilde{I}_d = \frac{1}{2}\int\{I_{xx}(u'')^2 + I_{yy}(v'')^2 + 2I_{xy}u''v'' + \Gamma(\theta'')^2 - 2\Gamma_x\theta''u'' - 2\Gamma_y\theta''v'' + C(\theta')^2\}\,\mathrm{d}z - \bar{P}U - \bar{Q}V - \bar{T}\Theta, \tag{29}$$

with $C = G(I_p - D)$ being the conventional St. Venant torsional stiffness factor.

In evaluating the variational equation $\delta\tilde{I}_d = 0$ we take account of the constraint boundary conditions $u(0) = v(0) = \theta(0) = u'(0) = \theta'(0) = 0$. The corresponding six conditions for the loaded end of the beams are the Euler boundary conditions

$$\begin{aligned} &I_{xx}u'' + I_{xy}v'' - \Gamma_x\theta'' = 0, && I_{xx}u''' + I_{xy}v''' - \Gamma_x\theta''' = -\bar{P} \\ &I_{xy}u'' + I_{yy}v'' - \Gamma_y\theta'' = 0, && I_{xy}u''' + I_{yy}v''' - \Gamma_y\theta''' = -\bar{Q} \\ &\Gamma_x u'' + \Gamma_y v'' + \Gamma\theta'' = 0, && \Gamma_x u''' - \Gamma_y v''' + \Gamma\theta''' - C\theta = -\bar{T}, \end{aligned} \tag{30}$$

for $z = L$, with the Euler differential equations being

$$\begin{aligned} I_{xx}u^{IV} + I_{xy}v^{IV} - \Gamma_x\theta^{IV} &= 0, \\ I_{xy}u^{IV} + I_{yy}v^{IV} - \Gamma_y\theta^{IV} &= 0, \\ -\Gamma_x u^{IV} - \Gamma_y v^{IV} + \Gamma\theta^{IV} - C\theta'' &= 0. \end{aligned} \tag{32}$$

In order to solve the problem as stated, we begin by obtaining from (31) the transformed differential equations

$$Ku^{IV} = (\Gamma_x I_{yy} - \Gamma_y I_{xy})\theta^{IV}, \quad Kv^{IV} = (\Gamma_y I_{xx} - \Gamma_x I_{xy})\theta^{IV}, \tag{32}$$

$$\Gamma_* \theta^{IV} - C\theta'' = 0, \tag{33}$$

where

$$K = I_{xx}I_{yy} - I_{xy}^2, \quad \Gamma_* = \Gamma - (\Gamma_x{}^2 I_{yy} - 2\Gamma_y\Gamma_x I_{xy} + \Gamma_x{}^2 I_{xx})/K, \tag{34}$$

and from (30) the transformed constraint boundary conditions

$$\begin{aligned} u'' = 0, \quad Ku''' &= -I_{yy}\bar{P} + I_{xy}\bar{Q} + (I_{yy}\Gamma_x - I_{xy}\Gamma_y)\theta''' \\ v'' = 0, \quad Kv''' &= -I_{xx}\bar{Q} + I_{xy}\bar{P} + (I_{xx}I_y - I_{xy}\Gamma_x)\theta''' \end{aligned} \tag{35}$$

$$\theta'' = 0, \qquad \Gamma_*\theta''' - C\theta' = -\bar{T} - (I_{yy}\Gamma_x - I_{xy}\Gamma_y)(\bar{P}/K) - (I_{xx}I_y - I_{xy}\Gamma_x)(\bar{Q}/K), \tag{36}$$

for $z = L$.

Equations (33) and (36), in conjunction with the conditions $\theta(0) = \theta'(0) = 0$, give as expression for θ,

$$\theta(z) = \frac{L}{C}\left[\bar{T} + \frac{\Gamma_x I_{yy} - \Gamma_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\bar{P} + \frac{\Gamma_y I_{xx} - \Gamma_x I_{xy}}{I_{xx}I_{xy} - I_{xy}^2}\bar{Q}\right]\left[\frac{z}{L} - \frac{\sinh\lambda L - \sinh\lambda(L-z)}{\lambda L\cosh\lambda L}\right], \lambda^2 = \frac{C}{\Gamma_*} \tag{37}$$

Having θ as in (37), we find u and v from eqns (32) and (35) in the form

$$\begin{aligned} u(z) &= \frac{\bar{P}I_{yy} - \bar{Q}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\left(\frac{Lz^2}{2} - \frac{z^3}{6}\right) + \frac{\Gamma_x I_{yy} - \Gamma_y I_{xy}}{I_{xx}I_{yy} - I_{yy}^2}\theta(z), \\ v(z) &= \frac{\bar{Q}I_{xx} - \bar{P}I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\left(\frac{Lz^2}{2} - \frac{z^3}{6}\right) + \frac{\Gamma_y I_{xx} - \Gamma_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\theta(z). \end{aligned} \tag{38}$$

We now use eqns (37) and (38) in order to obtain the values of $\Theta = \theta(L)$. $U = u(L)$. $V = v(L)$ in terms of $\bar{P}$, $\bar{Q}$, $\bar{T}$, so as to obtain from eqns (24), as *approximate* expressions for flexibility coefficients

$$\begin{aligned} C_{\Theta T} &= \frac{L}{C}\left(1 - \frac{\tanh\lambda L}{\lambda L}\right), \quad C_{\Theta P} = \frac{L}{C}\frac{\Gamma_x I_{yy} - \Gamma_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\left(1 - \frac{\tanh\lambda L}{\lambda L}\right) \\ C_{UP} &= \frac{L^3}{3}\frac{I_{yy}}{I_{xx}I_{yy} - I_{xy}^2} + \frac{L}{C}\left(\frac{\Gamma_x I_{yy} - \Gamma_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)^2\left(1 - \frac{\tanh\lambda L}{\lambda L}\right), \end{aligned} \tag{39}$$

etc. Introduction of these expressions into the defining relations (7) for the coordinates of the center of shear and of twist then give as approximations for these coordinates

$$\boxed{y_S = y_T = \frac{\Gamma_x I_{yy} - \Gamma_y I_{xy}}{I_{xx}I_{yy} - I_{yy}^2}, \quad x_S = x_T = -\frac{\Gamma_y I_{xx} - \Gamma_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}.} \tag{40}$$

We note that in the event that the x, y-axes in the cross section are principal elastic axes we have $I_{xy} = 0$, and eqn (40) reduces to the simplified form $x_s = -\Gamma_y/I_{yy}$, $y_s = \Gamma_x/I_{xx}$. For the case of a constant modulus E, these latter formulas agree with the results previously obtained in the classical Weber–Trefftz considerations, as well as with our earlier approximate results which followed from a determination of flexibility coefficients through use of the principal of minimum complementary energy in conjunction with approximations for stresses as given by the St. Venant theory of torsion and flexure[1].

An interesting special case of the above is the case of a solid circular cross section for which $\phi = 0$, throughout. We then have that the location of the shear center coincides with the elastic centroid of the cross section, for all possible variations of E, as long as it is assumed that G does not vary. This result should be compared with the well-known result for a uniform *semi*-circular cross section which may be interpreted as the case of a complete circular cross section with vanishing E *and* G over one-half of the section. For this case we have as the distance of the centroid from the straight portion of the cross sectional boundary curve $x_c = 4a/3\pi \approx 0.42a$, while at the same time the distance of the shear center is given by $x_s = 8a/5\pi \approx 0.51a$, with the difference in assumptions concerning the distribution of G for the two cases evidently being responsible for a significant effect on the location of the center of shear and of twist.

APPROXIMATE DETERMINATION OF STIFFNESS COEFFICIENTS THROUGH USE OF MINIMUM COMPLEMENTARY ENERGY PRINCIPLE

We now consider the use of the variational equation $\delta I_s = 0$ with I_s given in eqn (11a), with $\sigma_x = \sigma_y = \tau_{xy} = 0$, and with complementary energy density

$$B = \frac{1}{2}(\sigma^2/E + \tau_x^2/G + \tau_y^2/G) \tag{41}$$

where $\sigma \equiv \sigma_z$, $\tau_x \equiv \tau_{xz}$, $\tau_y \equiv \tau_{yz}$.

We have previously considered the application of this principle, for the case of cross sections with the x-axis an axis of symmetry, and with E independent of x and y[1], on the basis of stipulating a St. Venant distribution

$$I_{yy}\sigma = EQ(L-z)y, \tag{42}$$

$$\tau_x = Q(t_x + \chi_{,y}) + T\psi_{,y}, \quad \tau_y = Q(t_y - \chi_{,x}) - T\psi_{,x} \tag{43}$$

with χ and ψ as stress functions, with t_x and t_y as particular solutions of

$$I_{yy}(t_{x,x} + t_{y,y}) - Ey = 0, \tag{44}$$

and with χ, ψ, Q and T to be determined by the variational equation in conjunction with the boundary condition $\tau_x\,dy - \tau_y\,dx = 0$ along $f(x, y) = 0$.

It was found in[1], with the help of transformations of some complexity, that the result so obtained, comes out to be $x_s = x_t = -\Gamma_y/I_{yy}$ and an analogous outcome may be anticipated for the problem without an axis of symmetry, as long as the analysis is restricted by the assumption that E is independent of x and y.

In what follows we consider the derivation of a different approximate result, of independent interest and—insofar as our subsequent bound calculations are concerned—associated with slightly better results than would follow from the use of an equation equivalent to (42).

Our starting assumption is now, in place of eqn (42), the relation

$$\sigma = (L-z)(\tau_{x,x} + \tau_{y,y}) \tag{45}$$

where, as in (43), τ_x and τ_y are independent of z and subject to the constraint boundary condition

$$f(x, y) = 0; \quad \tau_x\,dy - \tau_y\,dx = 0. \tag{46}$$

An introduction of (45) and (46) into (41) and (11a) leaves, upon carrying out the integration with respect to z, the approximate energy expression

$$\bar{I}_s = \int\int [(\bar{U} - y\bar{\Theta})\tau_x + (\bar{V} + x\bar{\Theta})\tau_y - \frac{L^3}{6E}(\tau_{x,x} + \tau_{y,y})^2 - \frac{L}{2G}(\tau_x^2 + \tau_y^2)]\,\mathrm{d}x\,\mathrm{d}y. \tag{47}$$

The variational equation $\delta\bar{I}_s = 0$, with the stipulation that $\delta\tau_x\,\mathrm{d}y - \delta\tau_y\,\mathrm{d}x = 0$ for $f = 0$, has as Euler equations two differential equations for τ_x and τ_y, of the form

$$\begin{aligned} \frac{\tau_x}{G} - \frac{L^2}{3}\left(\frac{\tau_{x,x} + \tau_{y,y}}{E}\right)_{,x} &= \frac{\bar{U} - y\bar{\Theta}}{L}, \\ \frac{\tau_y}{G} - \frac{L^2}{3}\left(\frac{\tau_{x,x} + \tau_{y,y}}{E}\right)_{,y} &= \frac{\bar{V} + x\bar{\Theta}}{L}. \end{aligned} \tag{48}$$

Equations (48) may be simplified by deducing from them the relation

$$\left(\frac{\tau_x}{G}\right)_{,y} - \left(\frac{\tau_y}{G}\right)_{,x} = -2\frac{\bar{\Theta}}{L}, \tag{49}$$

which, in turn, implies as expressions for τ_x and τ_y in terms of an arbitrary function $\phi(x, y)$,

$$\tau_x = G(\phi_{,x} - \bar{\Theta}y/L), \quad \tau_y = G(\phi_{,y} + \bar{\Theta}x/L). \tag{50}$$

Introduction of eqns (50) into (48) then gives further

$$\begin{aligned} \phi_{,x} - \frac{L^2}{3}\left(\frac{(G\phi_{,x})_{,x} + (G\phi_{,y})_{,y}}{E}\right)_{,x} &= \frac{\bar{U}}{L} + \frac{L^2}{3}\left(\frac{xG_{,y} - yG_{,x}}{E}\right)_{,x}\frac{\bar{\Theta}}{L}, \\ \phi_{,y} - \frac{L^2}{3}\left(\frac{(G\phi_{,x})_{,x} + (G\phi_{,y})_{,y}}{E}\right)_{,y} &= \frac{\bar{V}}{L} + \frac{L^2}{3}\left(\frac{xG_{,y} - yG_{,x}}{E}\right)_{,y}\frac{\bar{\Theta}}{L}. \end{aligned} \tag{51}$$

While it is possible to continue the analysis for variable G, *the results of not doing this become sufficiently simpler to justify a restriction from here on to the case* G = *const.* With this restriction, and with observation of the condition $\int E\phi\,\mathrm{d}S = 0$, we readily obtain from (51), as a second order differential equantion for ϕ,

$$\phi - (GL^2/3E)\nabla^2\phi = \bar{U}x/L + \bar{V}y/L. \tag{52}$$

Equation (52) differs significantly from the corresponding equation for the theory of torsion and flexure in accordance with St. Venant by the appearance of the *first* term on the left. We note that for slender beams, with representative cross sectional dimension $a \ll L$ this term will be small compared to the second term, of relative order Ea^2/GL^2, and that considering the form of the differential equation, we may take account of this term by an iterative procedure. The physical reason for the occurrence of the first term in (52) is evidently the stipulation of a condition of no cross-sectional warping at the fixed end of the beam. While this condition is disregarded in the St. Venant formulation, it is taken account of, approximately, in the present approximate solution through use of the principle of minimum complementary energy.

In order to obtain approximate expressions for flexibility coefficients we now consider the solution of (52) to consist of three parts,

$$\phi = \bar{\Theta}\phi_\theta/L + (\bar{U}\phi_u/L + \bar{V}\phi_v/L)/G \tag{53}$$

with the functions ϕ_θ, ϕ_u, ϕ_v subject to the differential equations.

$$\left(\nabla^2 - \frac{3E}{GL^2}\right)(\phi_\theta, \phi_u, \phi_v) = -\frac{3E}{L^2}(0, x, y), \tag{54}$$

and to the boundary conditions

$$f = 0; \qquad \phi_{\theta,x}\,\mathrm{d}y - \phi_{\theta,y}\,\mathrm{d}x = y\,\mathrm{d}y + x\,\mathrm{d}x, \tag{55a}$$

and

$$f = 0; \quad (\phi_{u,x}, \phi_{v,x})\,\mathrm{d}y - (\phi_{u,y}, \phi_{v,y})\,\mathrm{d}x = 0. \tag{55b}$$

With ϕ_θ, ϕ_u and ϕ_v determined through eqns (54) and (55), we then have as expressions for cross sectional forces P, Q and torque T, on the basis of eqns (2) and (50),

$$PL = G\bar{\Theta}\int(\phi_{\theta,x} - y)\,\mathrm{d}S + \bar{U}\int\phi_{u,x}\,\mathrm{d}S + \bar{V}\int\phi_{v,x}\,\mathrm{d}S, \tag{56a}$$

with a corresponding expression for Q, and

$$TL = G\bar{\Theta}\int(x\phi_{\theta,y} - y\phi_{\theta,x} + x^2 + y^2)\,\mathrm{d}S + \bar{U}\int(x\phi_{u,y} - y\phi_{u,x})\,\mathrm{d}S + \bar{V}\int(x\phi_{v,y} - y\phi_{v,x})\,\mathrm{d}S. \tag{56b}$$

A comparison of eqns (56) with eqns (3) gives as approximate expressions for stiffness coefficients

$$\begin{aligned} K_{PU} &= L^{-1}\int\phi_{u,x}\,\mathrm{d}S, \quad K_{PV} = L^{-1}\int\phi_{v,x}\,\mathrm{d}S, \\ K_{P\Theta} &= GL^{-1}\int(\phi_{\theta,x} - y)\,\mathrm{d}S, \ldots \\ K_{TU} &= L^{-1}\int(x\phi_{u,y} - y\phi_{u,x})\,\mathrm{d}S, \ldots \\ K_{T\Theta} &= GL^{-1}\int(x\phi_{\theta,y} - y\phi_{\theta,y} + x^2 + y^2)\,\mathrm{d}S. \end{aligned} \tag{57}$$

STIFFNESS COEFFICIENTS EXPRESSED IN TERMS OF WARPING FUNCTION FOR ST. VENANT TORSION

It is convenient to designate the warping function for St. Venant torsion within the present context by $\phi_\theta^{(0)}$, with this function being the solution of the boundary value problem.

$$\nabla^2\phi_\theta^{(0)} = 0; \quad (\phi_{\theta,x}^{(0)}\,\mathrm{d}y - \phi_{\theta,y}^{(0)}\,\mathrm{d}x)_b = (x\,\mathrm{d}x + y\,\mathrm{d}y)_b. \tag{58}$$

We then have†

$$\int(\phi_{\theta,x}^{(0)} - y)\,\mathrm{d}S = \int(\phi_{\theta,y}^{(0)} + x)\,\mathrm{d}S = 0, \tag{59}$$

and

$$\int(x\phi_{\theta,y}^{(0)} - y\phi_{\theta,x}^{(0)} + x^2 + y^2)\,\mathrm{d}S = I_p - D = C. \tag{60}$$

We note, specifically, on the basis of eqns (59) that $\phi_\theta^{(0)}$ is associated with vanishing values of the coefficients $K_{P\Theta}$ and $K_{Q\Theta}$ and that therefore eqn (58) represents an inadequate approximation to the contents of eqns (54) and (55) insofar as the determinations of ϕ_θ is concerned. We resolve this difficulty by considering the improved approximation

$$\phi_\theta = \phi_\theta^{(0)} + \phi_\theta^{(1)}, \tag{61}$$

†See for example, *Love's Treatise*, 4th Edn, pp. 311–313 (1934).

with $\phi_\theta^{(1)}$ determined from the relations

$$\nabla^2\phi_\theta^{(1)} = \frac{3E}{GL^2}\phi_\theta^{(0)}, \quad (\phi_{\theta,x}^{(1)}\,\mathrm{d}y - \phi_{\theta,y}^{(1)}\,\mathrm{d}x)_b = 0. \tag{62}$$

We now obtain, upon observation of (62),

$$K_{P\Theta} = GL^{-1}\int \phi_{\theta,x}^{(1)}\,\mathrm{d}S = \frac{G}{L}\int\left[(x\phi_{\theta,x}^{(1)})_{,x} + (x\phi_{\theta,y}^{(1)})_{,y} - \frac{3E}{GL^2}x\phi_\theta^{(0)}\right]\mathrm{d}S$$

$$= \frac{G}{L}\oint x(\phi_{\theta,x}^{(1)}\,\mathrm{d}y - \phi_{\theta,y}^{(1)}\,\mathrm{d}x)\,\mathrm{d}s - \frac{3}{L^3}\int Ex\phi_\theta^{(0)}\,\mathrm{d}S = -\frac{3}{L^3}\int Ex\phi_\theta^{(0)}\,\mathrm{d}S, \tag{63a}$$

with a corresponding relation

$$K_{Q\Theta} = -3L^{-3}\int Ey\phi_\theta^{(0)}\,\mathrm{d}S, \tag{63b}$$

and with $K_{T\Theta}$ given, on the basis of (60) and (57), by

$$K_{T\Theta} = GL^{-1}C. \tag{63c}$$

In evaluating the remaining stiffness coefficients, we may use the approximations $\phi_u = \phi_u^{(0)}$ and $\phi_v = \phi_v^{(0)}$, with the boundary conditions (55b) and the differential equations $\nabla^2(\phi_u^{(0)}, \phi_u^{(0)}) = -EL^{-2}(x, y)$. With this we obtain

$$K_{PU} = L^{-1}\int \phi_{u,x}^{(0)}\,\mathrm{d}S = L^{-1}\int [(x\phi_{u,x}^{(0)})_{,x} + (x\phi_{u,y}^{(0)})_{,y} + 3EL^{-2}x^2]\,\mathrm{d}S$$

$$= L^{-1}\oint x(\phi_{u,x}^{(0)}\,\mathrm{d}y - \phi_{u,y}^{(0)}\,\mathrm{d}x) + 3L^{-3}\int Ex^2\,\mathrm{d}S = 3L^{-3}\int Ex^2\,\mathrm{d}S, \tag{64a}$$

and, analogously,

$$K_{PV} = K_{QU} = 3L^{-3}\int Exy\,\mathrm{d}S, \qquad K_{QV} = 3L^{-3}\int Ey^2\,\mathrm{d}S. \tag{64b}$$

Evidently, these results are such that the effect of transverse shear deformation is not included, and it is apparent that the calculations including this effect will depend on a consideration of functions $\phi_u^{(1)}$, $\phi_v^{(1)}$ in approximations $\phi_u = \phi_u^{(0)} + \phi_u^{(1)}$, etc. with $\nabla^2\phi_u^{(1)} = -(3E/GL^2)\phi_u^{(0)}$, etc.

It remains now to evaluate K_{TU} and K_{TV}, in such a way as to express these quantities in terms of integrals involving the function $\phi_\theta^{(0)}$, if possible. This is accomplished as follows. We now use, in the defining relation

$$K_{TU} = L^{-1}\int (x\phi_{u,y}^{(0)} - y\phi_{u,x}^{(0)})\,\mathrm{d}S, \tag{65}$$

Green's theorem, involving $\phi_u^{(0)}$ and $\phi_\theta^{(0)}$ and appropriate relations satisfied by these functions, in the form

$$\int (\phi_{\theta,x}^{(0)}\phi_{u,x}^{(0)} + \phi_{\theta,y}^{(0)}\phi_{u,y}^{(0)})\,\mathrm{d}S = \oint \phi_\theta^{(0)}(\phi_{u,x}^{(0)}\,\mathrm{d}y - \phi_{u,y}^{(0)}\,\mathrm{d}x) - \int \phi_\theta^{(0)}\nabla^2\phi_u^{(0)}\,\mathrm{d}S = 3L^{-2}\int Ex\phi_\theta^{(0)}\,\mathrm{d}S. \tag{66}$$

In order to see that the l.h.s. of (66) is in fact what we wish to have, in place of the r.h.s. in (65), we now make use, in place of $\phi_\theta^{(0)}$, of the associated torsion stress function $\Psi^{(0)}$, defined by $\Psi_{,y}^{(0)} = \phi_{\theta,x}^{(0)} - y$ and $\Psi_{,x}^{(0)} = -\phi_{\theta,y}^{(0)} - x$, and therewith by $\nabla^2\Psi^{(0)} = -2$ and $(\Psi_{,y}^{(0)}\,\mathrm{d}y + \Psi_{,x}^{(0)}\,\mathrm{d}x)_b = 0$.

We then have

$$\int(\phi_{\theta,x}^{(0)}\phi_{u,x}^{(0)}+\phi_{\theta,y}^{(0)}\phi_{u,y}^{(0)})\,dS=\int[(y+\Psi_{,y}^{(0)})\phi_{u,x}^{(0)}-(x+\Psi_{,x}^{(0)})\phi_{u,y}^{(0)}]\,dS$$

$$=\int(y\phi_{u,x}^{(0)}-x\phi_{u,y}^{(0)})\,dS+\int(\Psi_{,y}^{(0)}\phi_{u,y}^{(0)}-\Psi_{,x}^{(0)}\phi_{u,y}^{(0)})\,dS. \tag{67}$$

A second application of Green's theorem, now to the second integral on the right, and observation of the properties of $\Psi^{(0)}$ shows that this integral vanishes. Therewith, and with (66) and (65), we then have as expression for K_{TU} in terms of $\phi_\theta^{(0)}$,

$$K_{TU}=-3L^{-3}\int Ex\phi_\theta^{(0)}\,dS. \tag{68a}$$

An analogous reduction gives

$$K_{TV}=-3L^{-3}\int Ey\phi_\theta^{(0)}\,dS. \tag{68b}$$

We note from (68) and (63) that our stiffness coefficients do satisfy, as they should, the symmetry relations $K_{TU}=K_{P\Theta}$ and $K_{TV}=K_{Q\Theta}$, and we also note that we have previously used a similar transformation as in going from (65) to (68a), in expressing the approximate value for $x_S=x_T$ in [1] for cross sections symmetric about the x-axis in terms of St. Venant's torsional warping function.

Having eqns (60), (63), (64) and (68) for stiffness coefficients, we now see from eqns (8) for the coordinates of the centers of twist and of shear in terms of these coefficients, in conjunction with the defining relations (26), that upon identifying $\phi_\theta^{(0)}$ with the function ϕ in (26) the present approximate analysis by means of the principle of minimum complementary energy does in fact lead to the same eqns (40) for the location of the centers as obtained through use of the principle of minimum potential energy. It is, however, worth noting in this connection that while in the complementary energy calculations the function $\phi_\theta^{(0)}$ appears as a logical consequence of the analysis, it's corresponding appearance in the potential energy calculations depends on a fortuitous *ad hoc* assumption in the displacement approximation eqn (21), and that no agreement between the two types of results would have occurred if instead of defining ϕ in eqn (20) as St Venant's torsional warping function some other definition had been used.

SOME EXPLICIT BOUNDS FOR INFLUENCE COEFFICIENTS

A return to our consideration of bound relations in eqns (17)–(19) indicates that the set of approximate flexibility coefficients C, in eqns (37)–(39), is in fact also a set of lower bound coefficients C^L. In this connection, we particularly note the factor $1-(\lambda L)^{-1}\tanh\lambda L$ in eqns (39) which makes these coefficients smaller than they would be without this factor.

Furthermore, we may utilize the analysis is eqns (20)–(38) for the purpose of solving the analogous problem, with $U=\bar{U}$, $V=\bar{V}$, $\Theta=\bar{\Theta}$ as constraint conditions for $z=L$, instead of the conditions $P=\bar{P}$, $Q=\bar{Q}$, $T=\bar{T}$, and with I_d in (11b) replaced by I_d^* in (12), in such a way that the values of P, Q, T which occur in $2\bar{I}_d^*=\bar{U}P+\bar{V}Q+\bar{Q}T$ are the same combinations of loaded-end values of derivatives of u, v, θ as occur in the expressions for $\bar{P}$, $\bar{Q}$, $\bar{T}$ in eqns (30). It follows from this that ***the upper bound stiffness coefficients K^U in eqns (15a) are in fact the elements of a matrix K^U which is the inverse of the matrix C^L which is implied by eqns*** (37)–(39).

Having thus obtained an upper bound quadratic form for the coefficients K of the matrix $\mathbf{K}$, in accordance with eqns (15a) and (13), we next observe that our analysis in eqns (45)–(57) in of such nature as to make the elements K in eqns (57) effectively lower bound coefficients K^L, in accordance with eqns (15b) and (13). Evidently, eqns (53)–(57) no more than *enable* us to calculate these coefficients K^L. However, considering the form of the differential eqn (52) and the order of magnitude considerations leading from (54) to (58), (61), (62) and (64), we have that the explicit approximations for the coefficients K in eqns (63), (64) and (68) are in fact the values of the lower bound coefficients K^L, ***except for additive terms of relative order*** $(a/L)^2$.

We may associate this conclusion, with another one which will result from a consideration of the upper bound coefficients K^U obtained by inversion of the matrix C^L. This complementary conclusion is that *the upper bound coefficients K^U obtained in this manner also agree with the approximate coefficients in eqns* (63), (64) *and* (68), except for terms of relative order $(a/L)^2$. This being the case it is then possible to state that *the approximate values of the coefficients K in* (63), (64) *and* (68) *are in agreement with the exact values, except for terms of relative order* $(a/L)^2$. It follows then further that eqns (40) for the coordinates of the centers of twist and of shear represent the location of these centers—as defined by eqns (5) and (6), in association with the described mixed boundary value problem in three-dimensional linear elasticity theory—exactly, except for terms of relative order $(a/L)^2$. We may, if we wish, obtain improved bounds K^L, including terms of relative order $(a/L)^2$, by extending the calculations based on eqns (53)–(57) to the extent of determining, by iteration, the functions $\phi_\theta^{(0)}+\phi_\theta^{(1)}+\phi_\theta^{(2)}$, $\phi_u^{(0)}+\phi_u^{(1)},\phi_v^{(0)}+\phi_v^{(1)}$. We may also obtain improved bounds K^U by carrying out the analysis based on the displacement approximations (20) *without* imposing the constraint relations (28). In comtemplating such a program it must, however, be born in mind that the present analysis is based on the assumption of a medium for which the axial stress σ_z results in no lateral contraction effects in x, y-planes. Consideration of this lateral contraction effect would mean greatly increased complexity of the calculations leading to the values K^U. Such calculations may be expected to leave the first-approximation bound results unchanged, while at the same time being responsible for second-approximation bound results involving additional terms of relative order a/L as well as of order $(a/L)^2$.

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